ASSESSING THE VALIDITY OF SUBOPTIMAL 2-STATE CLOCK MODELS

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ABSTRACT: In real-time Kalman filter applications involving clocks, it is common practice to model the clock errors with simply two state variables, namely phase and frequency. It has been pointed out in previous PTTI papers that such models are suboptimal because they cannot accomodate flicker noise components exactly. Thus it is desirable to have a means of assessing the "goodness of fit" of a proposed 2-state model in the particular application at hand. The methodology suggested here involves first choosing a higher-order model which is used as a truth model for purpose of analysis. The suboptimal gains generated by the 2-state model being assessed are then cycled through the truth model to generate realistic error covariances to be associated with the suboptimal filter. This method of assessing the degree of suboptimality of the 2-state model is especially useful in prediction application where the Δt interval is relatively large. Subtleties and possible pitfalls in the design of the truth model are discussed in the paper, and examples are presented.

1. Introduction

It has been pointed out in previous PTTI papers[1,2] that flicker noise cannot be modeled with a finite-order state model. Thus one must accept some degree of suboptimality in the design of a Kalman filter for any clock where flicker noise is an appreciable part of the random process. The design objective then must be to choose a reasonable state model which yields a good approximation of the clock's random processes over the range of interest in the application at hand. For example, in a GPS receiver the measurement information for updating the clock comes at a fairly rapid rate, so the clock model can be rather crude. On the other hand, in applications where the clock must predict time over a considerable span, it might be necessary to have a more elaborate state model in the Kalman filter. In either event, it is important to be able to assess the validity of the medel for the application at hand, and this is the main thrust of this paper. The procedure for performing the assessment is to first develop a higher-order model which represents the clock's random processes faithfully over a wide range of parameters. This is considered to be the "truth model". The lesser-order design being considered for real-time implementation is then compared with the truth model. It will be seen presently that there are subtleties in making this comparison, and these will be discussed in the subsequent sections.

2. Methodology for development of truth model

Flicker Noise

Flicker noise process is defined to have a spectral density of $1/|\omega|$. Many ways have been proposed to model flicker noise process and generate a process having flicker noise-like characteristics. All methods are just approximations of the flicker noise process [3].

Let us first consider a shaping filter approach for generating a flicker noise process. If a white noise input is applied to a shaping filter whose transfer function is $1/\sqrt{s}$, then the output will have flicker noise characteristics. The impulse response of such filter becomes

$$h(t,t_0) = \frac{1}{\sqrt{\pi(t - t_0)}}$$

where t_0 is the initial time. Note that $h(t,t_0)$ is not decomposable into a product of arbitrary functions of $M(t)$ and $N(t_0)$ such that $h(t,t_0) = M(t)N(t_0)$. Thus according to linear system theory [4], the system cannot be realized by a finite-dimensional linear system. The dimension of the system must be infinite. Hence, any model of clock noises with finite-dimension must be an approximation.

In order to obtain a finite dimensional linear system for a flicker noise process, we will examine a Pade approximation method. This method can provide an approximate rational function of s for an irrational transfer function of the form $1/\sqrt{s}$.

Padé Approximation

The Padé approximation is such as to generate a rational fraction approximation to the value of a function f(s) at a point. It matches a formal series expansion as far as possible. Here "formal series" means that it may diverge when the function is expanded in series.

Consider a function $f(s) = 1/\sqrt{s}$ at $s = 1$. Its series expansion is assumed as follows:

$$f(s) = \frac{1}{\sqrt{s}}$$

$$= \sum_{k=0}^{\infty} C_k s^k \qquad (1)$$

Then, a one-sided [m,n] Padé approximant $R_{m,n}(s)$ for f(s) is defined as follows [5,6]:

$$R_{m,n}(s) = \frac{P_m(s)}{Q_n(s)} \qquad (2)$$

where $P_m(s)$ and $Q_n(s)$ are polynomials in s and are coprime to each other.
Or

$$R_{m,n}(s) = \frac{p_0 + p_1 s + p_2 s^2 + \dots + p_m s^m}{q_0 + q_1 s + q_2 s^2 + \dots + q_n s^n} \qquad (3)$$

Coefficients of $P_m(s)$ and $Q_n(s)$ are determined by

$$f(s) - R_{m,n}(s) = \mathcal{O}(s^{m+n+1}) \qquad (4)$$

where $\mathcal{O}$ is a 'big O'.

From equations (3) and (4), we have

$$(C_0 + C_1 s + C_2 s^2 + \ \ldots\ )(q_0 + q_1 s + q_2 s^2 + \ldots + q_n s^n) - (p_0 + p_1 s + p_2 s^2 + \ldots + p_m s^m) = \mathcal{O}(s^{m+n+1})$$

Equating coeff icients of like powers of s leads to a set of linear algebraic equations. For normalization, set $Q_n(0) = q_0 = 1$.
Then

$$\begin{aligned}
p_0 &= C_0 q_0 \\
p_1 &= C_1 q_0 + C_0 q_1 \\
p_2 &= C_2 q_0 + C_1 q_1 + C_0 q_2 \\
&\;\;\bullet \\
&\;\;\bullet \\
p_m &= C_m q_0 + C_{m-1} q_1 + \ \ldots\ + C_0 q_m \\
0 &= C_{m+1} q_0 + C_m q_1 + \ \ldots\ + C_{m-n+1} q_n \\
&\;\;\bullet \\
&\;\;\bullet \\
0 &= C_{m+n} q_0 + C_{m+n-1} q_1 + \ \ldots\ + C_m q_n
\end{aligned} \tag{5}$$

where $C_m = 0$ if $m < 0$ and $q_j = 0$ if $j > n$.
The solution to the above equation is uniquely determined provided that the Hankel matrix of the system is nonsingular [5]. In Table 1, partial entries of Padé approximants are listed.

It is known that, for accurate approximation over finite ranges of frequency, it is best to choose orders m and n of numerator and denominator polynomials to be equal, or nearly equal [6]. In linear system theory, a transfer function is realizable by a finite dimensional linear system, if and only if it is a proper rational function [4]. Thus, a typical choice of m and n to satisfy the realizability condition is that the order of denominator be greater than that of numerator, i.e., $n - m > 0$. We will mainly consider the case $n - m = 1$ among many possible rational approximants. This corresponds to superdiagonal entries in the Pade table. Elements in the table other than the diagonal and superdiagonal ones are unstable because of negative coefficients. Diagonal and superdiagonal entries have negative real poles and zeros; hence they can be realized as stable systems. They also have a minimum phase. But by choosing superdiagonal entries for realization, we can not only ensure the stability, but also finite output variance.

In Figures 1 and 2, frequency and unit-step time responses of diagonal and superdiagonal entries of Padé approximants are shown.

Finally, poles and zeros of diagonal and superdiagonal entries are given as follows:

$$\text{poles: } P_k = -\tan^2 \frac{(2k+1)\pi}{2(m+n+1)} \quad , \quad k = 0, 1, \ldots , [\frac{m+n-1}{2}] \tag{6}$$

$$\text{zeros: } Z_k = -\tan^2 \frac{k\pi}{(m+n+1)} \quad , \quad k = 1, 2, \ldots , [\frac{m+n}{2}]$$

where [r] implies the greatest integer $\leq$ r. Poles and zeros are located alternately on the negative real axis which is a branch cut of $1/\sqrt{s}$.

3. Cycling suboptimal gains through truth model

State model of clock noises

From Padé approximants, we get an approximate rational transfer function $R_{m,n}(s)$ for a realization of the flicker noise-like characteristic.

$$R_{m,n}(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0} \tag{7}$$

The output of $R_{m,n}(s)$ has an approximate flicker noise-like characteristic when the input is a white noise. It is assumed that the order m of the numerator polynomial is chosen to n-1 in order for the output variance to be finite.

The total clock noise can now be obtained as a sum of inherent clock noise processes, i.e., white noise, flicker noise, random-walk noise, and other higher-order or lower-order noise processes. In practice, it is often assumed that the clock error can be well represented by considering just three clock noises - white noise, flicker noise, and random-walk noise. Based on this assumption, we can draw a block diagram of clock noise model as shown in Figure 3. In Figure 3, the input white noises are independent of each other

The power spectral density (PSD) of $S_\Omega(\omega)$ of instantaneous fractional frequency deviation $\Omega(t)$ is given by

$$S_\Omega(\omega) = \frac{h_0}{2} + \pi h_{-1}\omega^{-1} + 2\pi^2 h_{-2}\omega^{-2} \tag{8}$$

where h_0, h_{-1}, and h_{-2} are the usual Allan variance parameters [7].

From Figure 3 and equation (7), we will now redraw the clock noise model as shown in Figure 4.

A state-space representation of the clock noise model based on Figure 4 can now be written as follows:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n+1} \\ \dot{x}_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & b_0 & b_1 & \cdots & b_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n+1} \\ x_{n+2} \end{bmatrix} + \begin{bmatrix} w_0 \\ w_{-2} \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ w_{-1} \end{bmatrix} \tag{9}
$$

It is convenient now to make a parallel decomposition of $R_{n-1,n}(s)$ in Figure 4. This leads to the block diagram shown in Figure 5. From Figure 5, we have a new state-space dynamic equation as follows:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n+1} \\ \dot{x}_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -\lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \vdots & \cdot & \cdot & \vdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n+1} \\ x_{n+2} \end{bmatrix} + \begin{bmatrix} w_0 \\ w_{-2} \\ K_1 w_{-1} \\ K_2 w_{-1} \\ \cdot \\ \cdot \\ \cdot \\ K_n w_{-1} \end{bmatrix} \tag{10}
$$

The advantage for doing this is twofold. First, the system configuration becomes less complicated. State variables may be assigned easily according to size of eigenvalues. Secondly, we may delete some state vectors which have very large or very small time constants, and thus these states are negligible in the overall system performance in a certain period of time. Or, we can identify important system modes and understand the system better. Hence, a dimensional reduction of the system may be easily achieved.

Determination of Kalman filter parameters

It is straightforward to obtain a Kalman filter from the state-space equation (10). For notation, refer to a reference [8]. State transition matrix $\Phi(\Delta t)$ is given by

$$\Phi(\Delta t) = \begin{bmatrix} 1 & \Delta t & \frac{1}{\lambda_1}(1-e^{-\lambda_1 \Delta t}) & \frac{1}{\lambda_2}(1-e^{-\lambda_2 \Delta t}) & \cdots & \frac{1}{\lambda_n}(1-e^{-\lambda_n \Delta t}) \\ 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & e^{-\lambda_1 \Delta t} & 0 & \cdots & 0 \\ 0 & 0 & 0 & e^{-\lambda_2 \Delta t} & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & \cdots & e^{-\lambda_n \Delta t} \end{bmatrix} \tag{11}$$

The covariance matrix Q for the discretized noise process w_k is given by

$$Q = \begin{bmatrix} Q_A & Q_B \\ Q_B^T & Q_D \end{bmatrix} \quad , \quad T = \text{transpose} \tag{12}$$

where $Q_A \in R^{(2x2)}$, $Q_B \in R^{(2xn)}$, and $Q_D \in R^{(nxn)}$

Each sub-matrix can now be obtained as follows:
For entries of sub-matrix Q_A, they are

$$Q_A(1,1) = \frac{1}{2}h_0\Delta t + 2h_{-1}(\Delta t)^2 + \frac{2\pi^2}{3}h_{-2}(\Delta t)^3 \tag{12.a}$$

$$\cong \frac{1}{2}h_0\Delta t + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{K_iK_j}{\lambda_i\lambda_j}\left\{ \Delta t - \frac{1-e^{-\lambda_i \Delta t}}{\lambda_i} - \frac{1-e^{-\lambda_j \Delta t}}{\lambda_j} + \frac{1-e^{-(\lambda_i+\lambda_j)\Delta t}}{\lambda_i+\lambda_j}\right\}\pi h_{-1} + \frac{2\pi^2}{3}h_{-2}(\Delta t)^3$$

$$Q_A(1,2) = \pi^2 h_{-2}(\Delta t)^2$$

$$Q_A(2,2) = 2\pi^2 h_{-2}\Delta t$$

For entries of Q_B,

$$Q_B(1,j) = \sum_{i=1}^{n} \frac{1}{\lambda_i} \left\{ \frac{1-e^{-\lambda_j \Delta t}}{\lambda_j} - \frac{1-e^{-(\lambda_i+\lambda_j)\Delta t}}{\lambda_i + \lambda_j} \right\} K_i K_j \pi h_{-1} \quad \text{for } 1 \le i, j \le n \quad (12.b)$$

$$Q_B(2,j) = 0$$

For entries of Q_D,

$$Q_D(i,j) = \frac{1-e^{-(\lambda_i+\lambda_j)\Delta t}}{\lambda_i + \lambda_j} K_i K_j \pi h_{-1} \qquad \text{for } 1 \le i, j \le n \qquad (12.c)$$

For a detailed derivation of $\Phi(\Delta t)$ and Q, see a reference [9].

Error Analysis and suboptimal filtering

Since our objective is to study error accumulation in the developed clock noise model, error analysis is needed using a Kalman filter. It was shown that the error covariance matrix can be propagated without forming state estimates in the Kalman filter equations. Hence, less steps can be taken in the Kalman filter equations by skipping the state estimations steps.

If accurate behavior of the flicker noise is essential, then the number of parallel blocks used to represent $1/\sqrt{s}$ may have to be fairly large. This is quite permissible in off-line analysis, and let its order be n. Then let us call this (n+2) dimensional system the "truth model" and from this truth model, construct a suboptimal filter by considering a subset of the system components of the truth model. Here, the suboptimal filter can be formed by retaining state vectors which represent significant modes of the system and discarding less significant ones.

In our study, the suboptimal filter considered was choosing the first two state vectors from the truth model state-space equation (10) with a different set of noises. It can be written as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad (13)$$

where w_1 and w_2 are white noise processes to be chosen appropriately.

In order to choose w_1 and w_2, observe the following facts. Random-walk noise is more divergent than the flicker noise. Actually, the flicker noise has a logarithmic divergence characteristic. Hence, the random-walk noise dominates the flicker noise in the long run. Since we are interested in the long-term behavior ($t \ge 1$) of clock error due to these three noise processes in the truth model, these facts can be utilized to choose noise components, w_1 and w_2, of the suboptimal filter, properly.

Definitely, the random-walk noise must be present. White noise may be retained to represent instantaneous fluctuations in the system. In addition, it is trivial to realize, thus the presence of the white noise does not add any complexity to the system. The flicker noise is deleted since its characteristic is in between the random-walk and white noise processes, and its effect will be absorbed to some extent in w_1 and w_2.

To assesserrors in the suboptimal filter, the truth model Kalman filter is run in parallel with the suboptimal filter. See Figure 6. There are two truth model Kalman filter loops. One is run with optimal gains and the other with suboptimal gains. The suboptimal filters produces suboptimal gain sequence when supplemented with zeros for the higher-order gains.

Optimal prediction error by Bode-Shannon method

The optimal prediction error variance for the clock phase $x_1(t)$ in the Kalman filter can be obtained by projecting ahead N steps in the truth model. However, the Bode-Shannon method also provides an alternative way of obtaining an analytical expression for the projection error. Furthermore, it does not involve any approximations, as is the case with the truth model used in the Kalman filter analysis. For detail, see a reference [10]. Here, we present only results:

$$\text{Optimal prediction error} = \frac{1}{2}h_0\,\Delta t + \pi h_{-1}(\Delta t)^2 + \frac{2}{3}\pi^2 h_{-2}(\Delta t)^3 \tag{14}$$

From the expression, we can clearly see that the random-walk noise eventually dominates the other two. For instance, with same input noise powers in three noise processes and $\Delta t = 1$, the random-walk phase noise becomes 10 times larger than the flicker phase noise only after 18 steps. One thing to note is that opt imal prediction error expression coincides with that of $Q_A(1,1)$ in equation (12.a).

4. Numerical Example

In this section, we will study clock error propagation in the truth model and 2-state suboptimal models with different Q matrices. For simplicity, we select a 5-state truth model which consists of white noise, random-walk noise, and third-order Pade approximant $R_{2,3}$ for flicker noise. Time response of $R_{2,3}(s)$ in Figure 2 closely follows the flicker noise process in the time interval, approximately from 0.07 to 13.9 seconds, corresponding to eigenvalues { -0.0718, -1, -13.9282 }. Time step size is chosen to be $\Delta t = 1$ which belongs to the interval.

For initialization of Kalman filter, we assume that a priori estimate $x_0^- = 0$, and a priori error covariance $P_0^- = 0$. With regard to the measurement noise, we set $R_k = 0.625\times10^{-17}$ in both the 5-state and 2-state suboptimal models. To calculate Q matrices, the following h_i prarmeters were used [1]:

$h_0 = 9.43 \times 10^{-20}$, $h_{-1} = 1.8 \times 10^{-19}$, and $h_{-2} = 3.8 \times 10^{-21}$

5-state truth model

The truth model state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\lambda_1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 \\ 0 & 0 & 0 & 0 & -\lambda_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} w_0 \\ w_{-2} \\ K_1 w_{-1} \\ K_2 w_{-1} \\ K_3 w_{-1} \end{bmatrix} \tag{15}$$

where

$$K_1 = 3.57265 \times 10^{-1},\ K_2 = \frac{2}{3},\ K_3 = 4.97607$$
$$\lambda_1 = 7.17967 \times 10^{-2},\ \lambda_2 = 1,\ \lambda_3 = 13.9282$$

Then, the state transition matrix Φ with $\Delta t = 1$ is obtained from equation (11).

$$\Phi = \begin{bmatrix} 1 & 1 & 0.9649 & 0.6321 & 0.0718 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.9307 & 0 & 0 \\ 0 & 0 & 0 & 0.3679 & 0 \\ 0 & 0 & 0 & 0 & 8.934 \times 10^{-6} \end{bmatrix}$$

The Q matrix is obtained from equation (12), and using the assumed h_0, h_{-1}, h_{-2}.

$$Q = \begin{bmatrix} 4.3067 & 0.3747 & 1.453 & 1.611 & 0.5027 \\ 0.3747 & 0.7501 & 0 & 0 & 0 \\ 1.453 & 0 & 0.6724 & 0.8628 & 0.7231 \\ 1.611 & 0 & 0.8628 & 1.088 & 1.266 \\ 0.5027 & 0 & 0.7231 & 1.266 & 5.097 \end{bmatrix} \times 10^{-19}$$

2-state suboptimal model with different Q matrices

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{13}$$

State transition matrix is identical with upper left (2x2) submatrix of the 5-state truth model. Determination of Q matrix in the 2-state suboptimal model is crucial to closely follow the behavior of the truth model. Four candidates for the Q matrix will be selected and compared each other for error propagation. This may aid in the selection of a proper Q matrix which best fits the suboptimal model.

Four Q matrix candidates are chosen as follows:

candidate 1: Upper left (2x2) submatrix of the truth model Q matrix in equation (12.a) is adopted.

$$Q = \begin{bmatrix} \frac{1}{2}h_0\Delta t + 2h_{-1}(\Delta t)^2 + \frac{2\pi^2}{3}h_{-2}(\Delta t)^3 & \pi^2 h_{-2}(\Delta t)^2 \\ \pi^2 h_{-2}(\Delta t)^2 & 2\pi^2 h_{-2}\Delta t \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} 4.306 & 0.3747 \\ 0.3747 & 0.7501 \end{bmatrix} \times 10^{-19}$$

candidate 2: It is formed by considering white noise and random-walk noise for w_1 and w_2, respectively. Since flicker noise is less divergent than random-walk noise and usually complex to represent. Hence, the flicker noise processs is excluded.

$$Q = \begin{bmatrix} \frac{1}{2}h_0\Delta t + \frac{2\pi^2}{3}h_{-2}(\Delta t)^3 & \pi^2 h_{-2}(\Delta t)^2 \\ \pi^2 h_{-2}(\Delta t)^2 & 2\pi^2 h_{-2}\Delta t \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} 0.7215 & 0.3747 \\ 0.3747 & 0.7501 \end{bmatrix} \times 10^{-19}$$

candidate 3: In reference [1], a different Q matrix is suggested for the same 2-state suboptimal model.

$$Q = \begin{bmatrix} \frac{1}{2}h_0\Delta t + 2h_{-1}(\Delta t)^2 + \frac{2\pi^2}{3}h_{-2}(\Delta t)^3 & 2h_{-1}\Delta t + \pi^2 h_{-2}(\Delta t)^2 \\ 2h_{-1}\Delta t + \pi^2 h_{-2}(\Delta t)^2 & \frac{1}{2\Delta t}h_0 + 2h_{-1} + \frac{8\pi^2}{3}h_{-2}\Delta t \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} 4.322 & 3.975 \\ 3.975 & 5.072 \end{bmatrix} \times 10^{-19}$$

candidate 4: In reference [2], another form of Q matrix is reported.

$$Q = \begin{bmatrix} \frac{1}{2}h_0\Delta t + 2h_{-1}(\Delta t)^2 + \frac{2\pi^2}{3}h_{-2}(\Delta t)^3 & 2h_{-1}\Delta t + \pi^2 h_{-2}(\Delta t)^2 \\ 2h_{-1}\Delta t + \pi^2 h_{-2}(\Delta t)^2 & 2\pi^2 h_{-2}\Delta t \end{bmatrix} \tag{19}$$

$$= \begin{bmatrix} 4.322 & 3.975 \\ 3.975 & 0.7501 \end{bmatrix} \times 10^{-19}$$

With these different Q matrices, Kalman filters of the 5-state and 2-state suboptimal models are run in parallel, as shown in Figure 6.

Discussion

Estimation error standard deviation for 170 steps have been plotted in Figure 7. From $t = 0$ to $t = 49\Delta t$ with step size $\Delta t = 1$, it is assumed that no measurement is available for 'free running' purpose. From $t = 50\Delta t$ to $t = 69\Delta t$, measurement starts for estimation and R_k is set to 0.625×10^{-17}. It shows that, after several steps, the standard deviations rapidly converge to a constant value. At $t = 70\Delta t$, filters are again in the free running mode for the next 100 steps.

From Figure 7, we may say that every candidate of the 2-state suboptimal model shows no big departure from the truth model except the third one among four Q matrix candidates. The third one is slightly larger, but not noticeable in the plot. This is expected since Q(1,1) of the Q matrix in equation (12.a) has the same formula as each candidate. In Figure 8, the suboptimal error analysis is performed with a 6-state truth model. This consistently agrees with results of the 5-state truth model.

Prediction error standard deviations of both the 5- and 6-state truth models do not show much departure from each other in case of time step size $\Delta t = 1$. See Figures 9 and 10. Deviation of prediction errors from the optimal one obtained by the Bode-Shannon method is about 30 % greater at the prediction of 80 steps.

In the Q matrices for 2-state suboptimal model, all four candidates except the third have same prediction errors and closely follow the truth model as in the estimation case. In the third candidate, the prediction error becomes slightly larger than any others after 80 steps. Only in the second candidate, the prediction error is slightly less because of the absence of the flicker noise component in the Q matrix. This tells us that the long-term effect due to the flicker noise is negligible in most cases ($t \geq 1$).

5. Conclusions

We have demonstrated that a higher-order truth model can be used off-line to assess the validity of a lower-order which is intended to be used on-ine. One should be careful to note though that no model of finite order can be expected to represent flicker noise over all possible ranges of parameter. Thus, one has to tailor the reference or truth model to the application at hand.

The time domain approach is especially useful in assessing prediction errors. This is because the $1/\sqrt{s}$ transfer function has a perfectly well-behaved unit-step response. Thus, if the approximate model response follows the ideal response closely over the prediction interval, one can be fairly confident that the approximate model will give good results for that particular prediction interval.

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Table 1. Pade approximants $R_{m,n}(s)$ of $\frac{1}{\sqrt{s}}$ at $s = 1$.

m \ n	1	2	3	4
0	$\frac{1}{1}$	$\frac{2}{s+1}$	$\frac{8}{-s^2+6s+3}$	$\frac{16}{s^3-5s^2+15s+15}$
1	$\frac{-s+3}{2}$	$\frac{s+3}{3s+1}$	$\frac{4s+4}{s^2+6s+1}$	$\frac{40s+24}{-s^3+15s^2+45s+5}$
2	$\frac{3s^2-10s+5}{8}$	$\frac{-s^2+10s+5}{20s+4}$	$\frac{s^2+10s+5}{5s^2+10s+1}$	$\frac{6s^2+20s+6}{s^3+15s^2+15s+1}$
3	$\frac{-5s^3+21s^2-35s+35}{16}$	$\frac{s^3-7s^2+35s+35}{56s+8}$	$\frac{-s^3+21s^2+105s+35}{70s^2+84s+6}$	$\frac{s^3+21s^2+35s+7}{7s^3+35s^2+21s+1}$

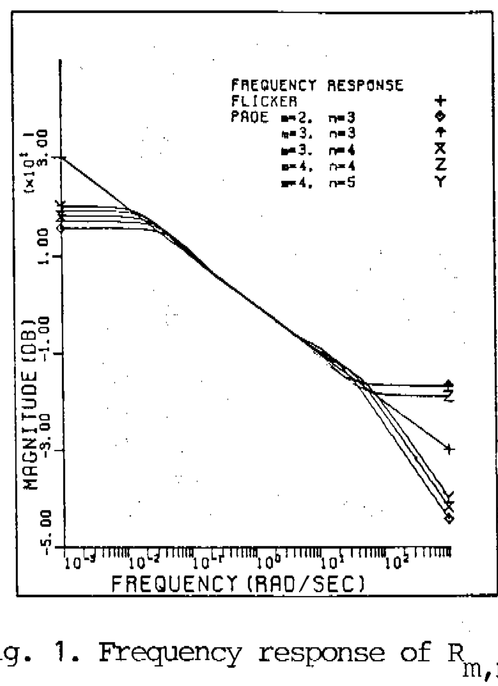

Fig. 1. Frequency response of $R_{m,n}(s)$

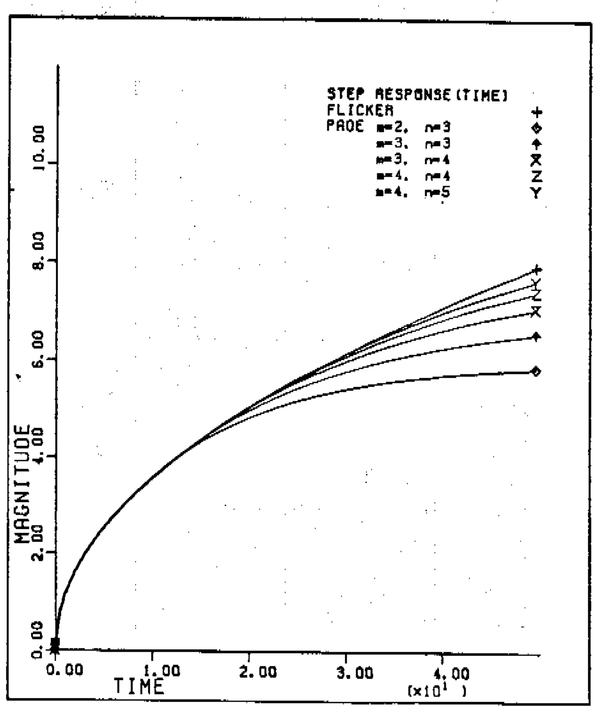

Fig. 2. Unit-step time response of $R_{m,n}(s)$

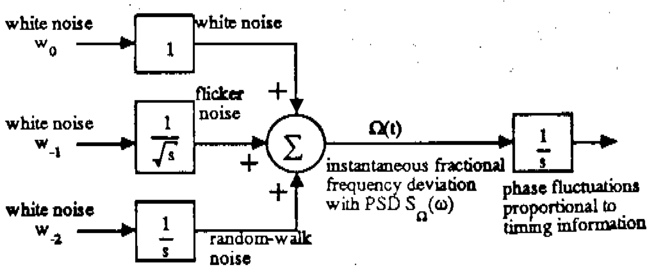

Fig. 3. A clock noise model

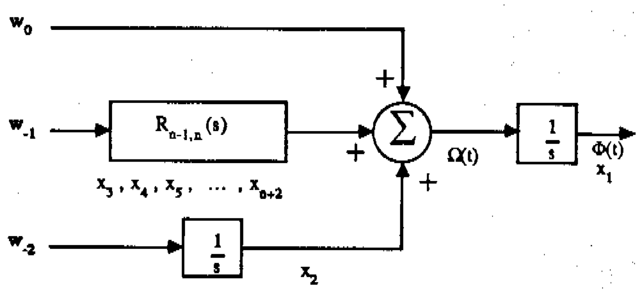

Fig. 4. A block diagram of the clock noise model

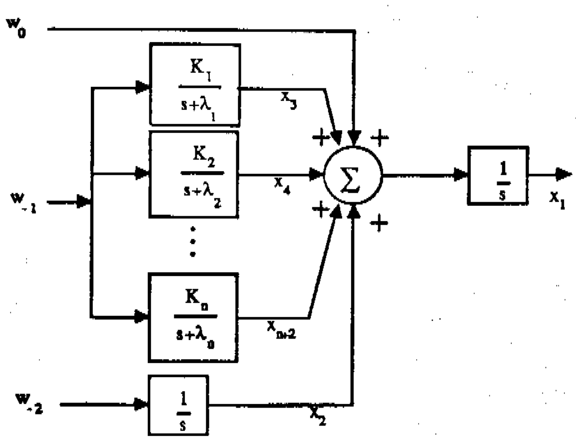

Fig. 5. Block diagram of parallel realization of the clock noise model

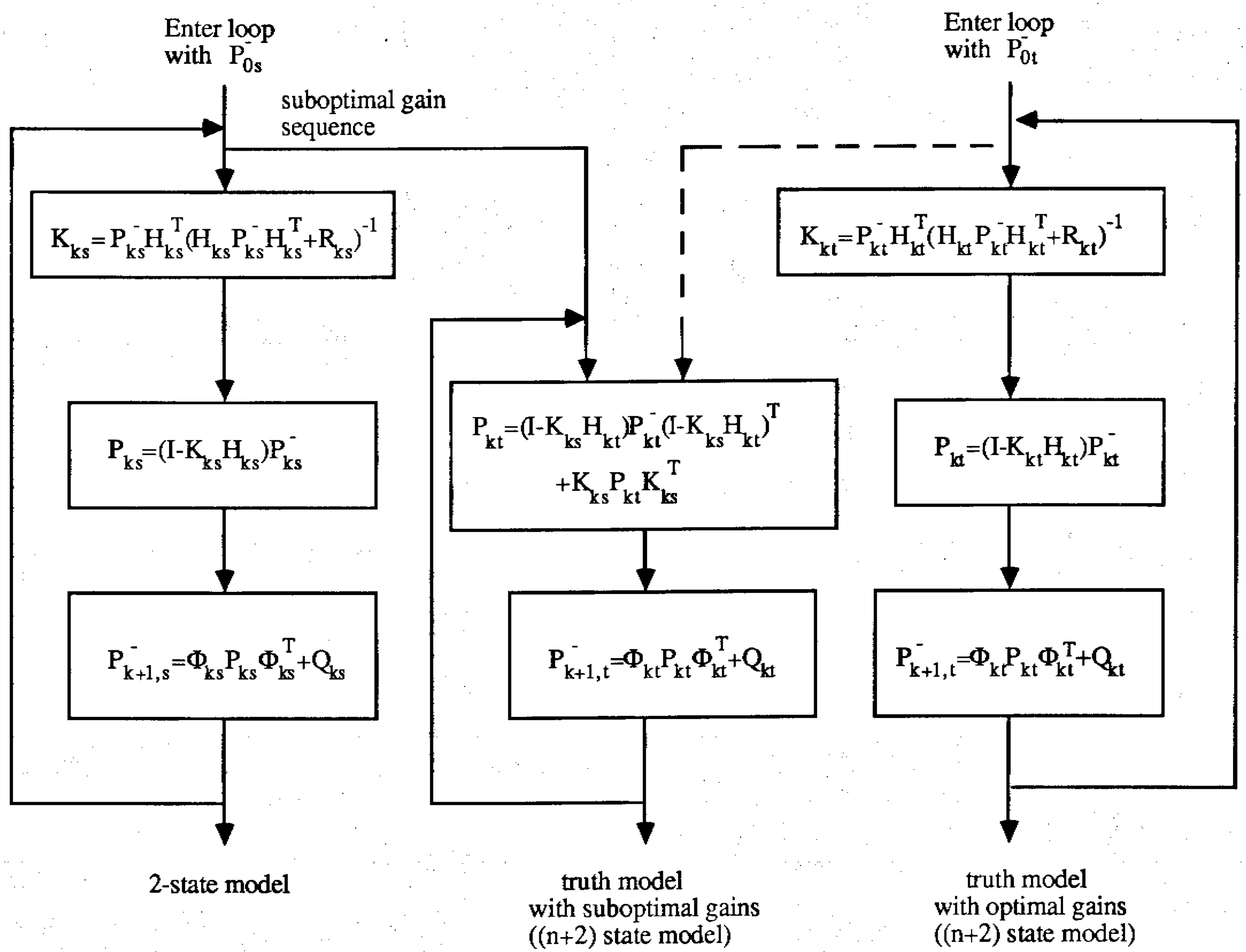

Fig. 6. Recursive loop for suboptimal error analysis

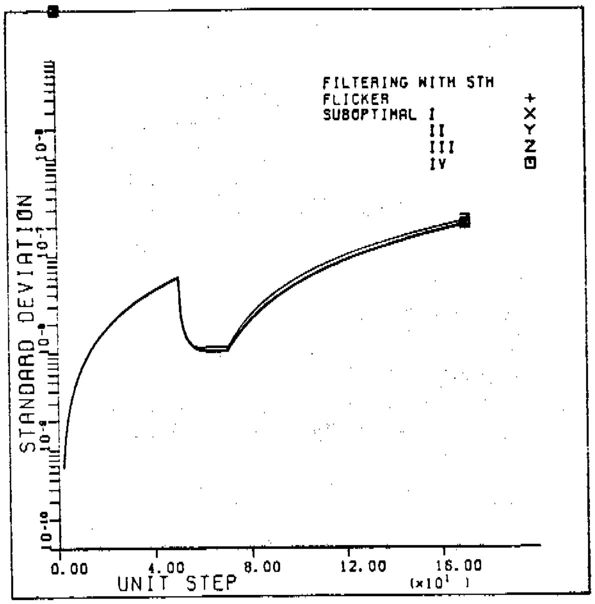

Fig. 7. Suboptimal error analysis with 5-state truth model (estimation)

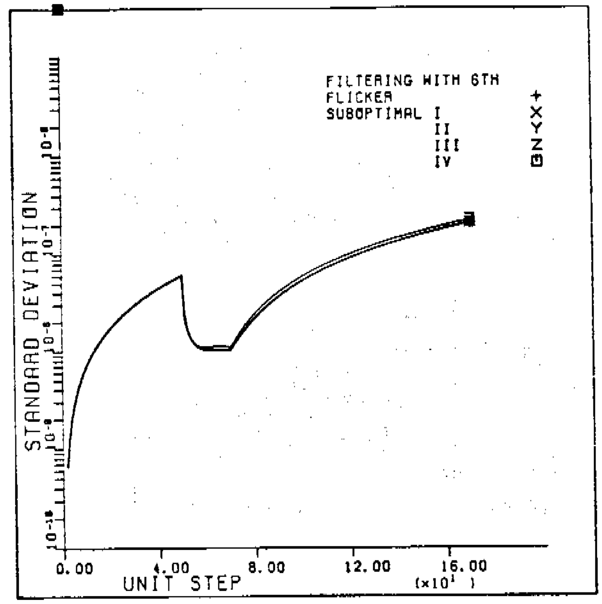

Fig. 8. Suboptimal error analysis with 6-state truth model (estimation)

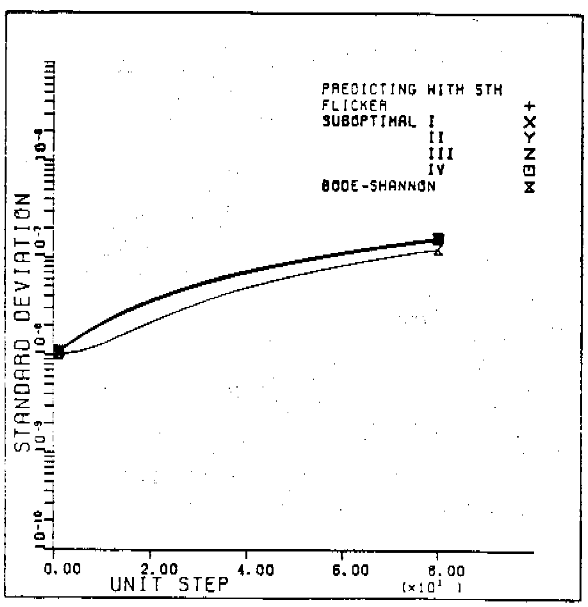

Fig. 9. Suboptimal error analysis with 5-state truth model (prediction)

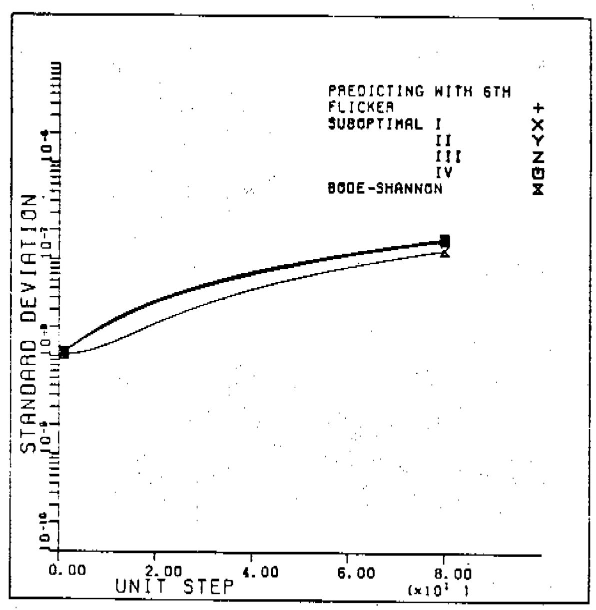

Fig. 10. Suboptimal error analysis with 6-state truth model (prediction)

QUESTIONS AND ANSWERS

MARK WEISS, NATIONAL BUREAU OF STANDARDS: I find this very interesting. It seems, if I understand it correctly, that you can use a two state Kalman filter with no flicker noise parameters included and get as good as a truth model.

MR. AHN: It depends on your time interval. If the step size is bigger than one, then the flicker noise is less dominant than the random walk noise process. As time goes on, the random noise process is more dominant than flicker noise. Then we just consider the two state model without the flicker noise process.

DAVID ALLAN, NATIONAL BUREAU OF STANDARDS: The comparison here on the last chart is a little deceptive because on a log plot the comparison in the worst region is nearly a factor of two different. The variation between the sub-optimum and the truth model is nearly a factor of two.

MR. AHN: This one is not realizable., it is just theoretical.

MR. ALLAN: I understand.

MR. AHN: There is no big difference between the truth model and the two state sub-optimal model if you choose the matrix properly.

MR. ALLAN: I guess that what I am trying to say is that optimum prediction for white frequency modulation and random walk frequency modulation is a very simple algorithm. You can approximate this ideal very closely. Better than is shown here.

MR. AHN: That is right.